Networked predictive control for nonlinear systems with stochastic disturbances in the presence of data losses

Shuang Li,[1] Guo-Ping Liu [2]

[1] School of Statistics, Xi'an University of Finance and Economics, Xi'an, 710100, China.

[2] School of Engineering, University of South Wales, Pontypridd, CF37 1DL,UK.

Abstract

Networked control allows monitoring a plant from a remote location through a communication channel and owns several attractive advantages. One of the major challenges is the control problem of stochastic nonlinear systems with packet losses and/or communication delays. In this paper, the networked control of nonlinear systems with stochastic disturbances in presence of packet losses is investigated. In order to reduce the effect of data packet losses on the system stability, a model predictive control method is proposed to compensate the packet losses in communication channel. By using stochastic stability theory and a previously designed Lyapunov controller, pth moment practical stability of the networked control system (NCS) is discussed, and a sufficient condition guaranteeing the practical stability of the closed-loop system is provided. Based on the sufficient condition, the relation formula between any prior given control target and the corresponding maximum time of consecutive packet losses is derived, and it is found that the ultimate bound of pth moment is mainly dependent on the maximum time of consecutive packet losses. As an example, networked control of the nonlinear chaotic Lorenz system with stochastic disturbances and data packet losses is considered to verify the effectiveness of the proposed method.

1 Introduction

With the development of computer and communication techniques, today we are more and more connected by networks, for example Ethernet, Internet, telephone network and so on, which make us exchange information like face to face though our distance is far. Based on these communication networks, some new research directions are generated, and one of them is networked control system. A networked control system (NCS) is a feedback system whose sensors, actuators, and controllers are interconnected via the communication network. A set of control sequence and output measurements can be transmitted from one location to another virtually at the same time. Compared to traditional point-to-point wiring systems, this type of control system has attractive advantages such as reducing the cost, easy diagnosis and maintenance, and improving the agility. It is not surprising that NCS has received increasing attention during the past period of more than 10 years [1–7]. However, since the limited bandwidth of communication networks and the networks shared by many users simultaneously, some undesirable phenomena such as data packet losses, signal transmission delays and so on inevitably occur, which often degrade the system performance and even lead to instability. Consequently, it has been an important content of NCS to overcome (or reduce) the impact of packet losses and time delays on the control performance. So far, several methods have been proposed for NCS in presence of data packet losses and/or transmission delays, for example, optimal control techniques [8–10], switched system approaches [11, 12], model predictive control methods [13–21], etc. In order to control the system actively when there are packet losses or delays, Liu et al. put forward a model-based compensation scheme that doesn't need to be known the probability distribution of data losses or delays in advance, but simply to suppose that there exists an upper bound of consecutive packet losses or delays. Because of the advantages of less restriction and easy operation, the model-based compensation schemes have attracted many scholars' attention, and been applied to a series of researches.

Up to now, most existing control techniques are concerned with linear systems. It is well known that actual systems are generally nonlinear, and include some stochastic factors inevitably. These stochastic factors may represent the inaccuracies of model parameter identification, or the interferences of the external factors. Some literatures discussed the stability of networked control of nonlinear systems within a deterministic framework through assuming the model's uncertainty to be a bounded sector. In [15], a robust control scheme combining model predictive control with a network delay compensation strategy was proposed to cope with model uncertainty, time-varying transmission delays, and packet dropouts . In [16], a Lyapunov-based model predictive control method was presented to study the stability of NCS under data losses for nonlinear systems with deterministic uncertain disturbances. For the uncertain large-scale nonlinear systems subject to asynchronous and delayed state feedback, an iterative distributed model predictive control was addressed [19]. However the knowledge on statistical modeling suggests that it seems more reasonable to consider a model with stochasticity rather than with deterministic uncertainty in that the full identification of nonlinear models is dependent on whether the residual between the model and data is a white noise or not. Therefore, the study on network control of stochastic nonlinear systems is necessary and important in practical applications. Through introducing Bernoulli distributions to model the phenomenon of missing measurements, Hu et al. investigated the finite-horizon filter design for a class of nonlinear time-varying systems with multiplicative noises and quantisation effects [23]. Wang et al. designed the quantized H_∞ controllers for a class of nonlinear stochastic time-delay network-based systems with probabilistic data missing [22]. It should be noticed that because of the theoretical difficulties of nonlinearity and stochasticity, the research on the networked control of stochastic nonlinear systems is still in the infant stage, especially for that by using the model-based compensation scheme. This article tries to shorten the gap by the model-based compensation strategy.

Motivated by the above-mentioned discussions, this paper discusses the networked control problem of continuously nonlinear systems with stochastic disturbances in presence of packet losses. First, based on a previously designed Lyapunov controller, the compensation strategy of data packet losses is presented.

Then, pth moment pratical stability of NCS with stochastic disturbance is investigated by using stochastic stability theory together with the proof of thought of [16] on nonlinear NCS with deterministic uncertainty. It shows that the ultimate boundness of pth moment of the trajectories is mainly influenced by the maximum time of consecutive packet losses, and a prior given control target can be achieved provided that the maximum time of consecutive packet losses is in some reasonable range. The obtained result is different from that of the deterministic uncertain case of [16], which implies that people should pay attention to the important differences between stochastic uncertainty and deterministic one in the stage of modeling or model identification so that a proper uncertain model is chosen. Finally, networked control of chaotic Lorenz system with stochastic disturbances in presence of packet losses is considered, and numerical simulations are presented to verify the effectiveness of this method.

2 Preliminaries

Throughout this paper, we use the following notations. If A is a vector or a matrix, its transpose is denoted by A^T. For a vector x,$\|x\| = \sqrt{x^T x}$ denotes the Euclidean norm of x. For a matrix $A = (A_{ij})_{n\times m}$,$\|A\| = \sqrt{\sum\limits_{i=1}^{n}\sum\limits_{j=1}^{m} A_{ij}^2}$ denotes the Frobenius norm of A,and $Tr(A)$ denotes the trace of the matrix A.If x is a random variable, $E(x)$ denotes the expectation of x. For a number a,$|a|$ denotes the absolute value of a. For the functions $f(x)$ and $h(x)$, $L_f h = \frac{\partial h}{\partial x} f(x)$ denotes the Lie derivative of the function h on f.

We consider the networked control problem of the following nonlinear systems with stochastic disturbances

$$dx(t) = F(x(t), u(t))dt + l(x(t))\Sigma d\omega(t) \tag{1}$$

where $x(t) \in \mathrm{X} \in R^n$ is the state, $u(t) \in U \in R^p$ is the controller, $\omega(t)$ is m-dimensional standard Wiener process and $\Sigma \in R^{m\times m}$ is its intensity matrix, $F(x,u) \in R^n$,$l(x) \in R^{n\times m}$, the origin 0 is an equilibrium point, namely $F(0,0) = l(0) = 0$ and $\{0\} \subset \mathrm{X}$. We assume that F and l are sufficiently smooth so that the existence, uniqueness and continuity of the solution hold. Also, there exists a global attractor (equilibrium point, periodic orbit, chaotic attractor,etc) $\Omega^0 \subset \mathrm{X}$ in the uncontrolled system and the set X can be the attraction domain of the attractor Ω^0.

Let $V(x)$ denote a candidate Lyapunov function which is twice continuously differentiable in x . There exists a number $r_1 > 0$ such that the set $\Omega_{r_1} = \{x : V(x) \le r_1\} \subset \mathrm{X}$ and $\Omega_{r_1} \supset \Omega^0$. $u = h(x)$ is a Lyapunov based controller satisfying the following conditions:

$$L_F V(x) + \frac{1}{2} Tr\left\{\Sigma^T l^T(x)\frac{\partial V^2}{\partial x^2} l(x)\Sigma\right\} \le -\rho V(x), (\rho > 0) \tag{2}$$

and

$$L_F V(x) \le -\rho V(x) \tag{3}$$

Remark 1 *In the system (1) if $F(x,u) = f(x) + g(x)u$, then a controller satisfying (2) and (3) can be constructed as*

$$h(x)=\begin{cases} 0, \quad if\ L_gV(x)=0 \\ -\frac{\psi+\sqrt{\psi^2+(L_gV(L_gV)^T)^2}}{L_gV(L_gV)^T}(L_gV)^T,\ else \end{cases} \tag{4}$$

where $\psi = L_fV(x) + \frac{1}{2}\left|Tr\left\{\Sigma^T l^T(x(t))\frac{\partial V^2}{\partial x^2}l(x(t))\Sigma\right\}\right| + \rho V(x)$ $(\rho > 0)$. *The above controller is inverse optimal, that is to say, which is optimal with respect to a meaningful cost functional. Readers can refer to [24–26] for more detailed discussions on inverse optimality.*

Assumption 1

(1)There exists $k_1 > 0$ such that $\forall x \in \mathrm{X}, y \in \mathrm{X}, u \in U$, $\|F(x,u) - F(y,u)\| \le k_1\|x-y\|$.

(2)There exist $M_1 > 0, M_2 > 0$ such that $\forall x \in \mathrm{X}, u \in U$, $\|F(x,u)\| \le M_1$ and $\|l(x(t))\Sigma\| \le M_2$.

(3)There exist $\eta_\Psi > 0, \eta_\varphi > 0$ such that $\forall x \in \mathrm{X}, y \in \mathrm{X}, u \in U$, $||\Psi(y,u) - \Psi(x,u)|| \le \eta_\Psi||y-x||$ and $||\varphi(y,u) - \varphi(x,u)|| \le \eta_\varphi||y-x||$, where $\Psi(x,u) = L_FV(x) + \frac{1}{2}Tr\left\{\Sigma^T l^T(x)\frac{\partial V^2}{\partial x^2}l(x)\Sigma\right\}$ and $\varphi(x,u) = L_FV(x)$.

(4) There exists $\lambda > 0$ such that $\forall x \in \mathrm{X}, y \in \mathrm{X}$, $||V(x) - V(y)|| \le f_V(||x-y||)$, where $f_V(||x-y||) = \lambda||x-y||$.

Remark 2 *The above inequalities are some basic assumptions for the stability study(e.g., Lipshitz property and boundedness).*

3 Networked predictive control for nonlinear systems with stochastic disturbance in the presence of data losses

Let us consider the problem that the system (1) is controlled by a communication network (see Fig.1). For NCS, we suppose that (1) the sensor is clock-driven, the controller and actuator are event-driven; (2) the information exchanged between the controller and actuator (or between the sensors and controller) can be a packet of data rather than a single value [27,28]; (3) the control signal $u(t)$ is implemented in a sample-and-hold fashion(zero-order hold), namely $u(t) = u(t_k)$ for $\forall t \in [t_k, t_{k+1})$ where $t_k = t_0 + k\Delta, k = 1, 2, ...,$ and Δ is a fixed time interval.

We introduce an auxiliary random variable $\theta(t_k)$ to characterize whether the data losses happen at the sampling instant t_k. When $\theta(t_k) = 1$, it means that the full states are available for the controller and the actuator receives the new control signals at the sampling instant t_k. When $\theta(t_k) = 0$, it means that either the full states are not available for the controller, or the actuator doesn't receive the new control signals at the sampling instant t_k. For the convenience of discussion, we suppose $\{t_{k_j}|j = 1, 2, ...\}$ denotes the set of asynchronous instants that the full states are available for the controller and the actuator receives the new control signals at the sampling instant t_k, where $t_{k_j} = t_0 + k_j\Delta$, and $k_{j'} > k_j$ for $\forall j' > j$. Different from some studies that discuss the stability of NCS based on the probability distribution of $\theta(t_k)$, in the present paper we only assume that there exists an upper bound $N_D > 1$ such that $\max\{k_{j+1} - k_j|j = 1, 2, ...\} \le N_D$. This implies that the maximum time of consecutive data packet losses is not more than $(N_D - 1)\Delta$.

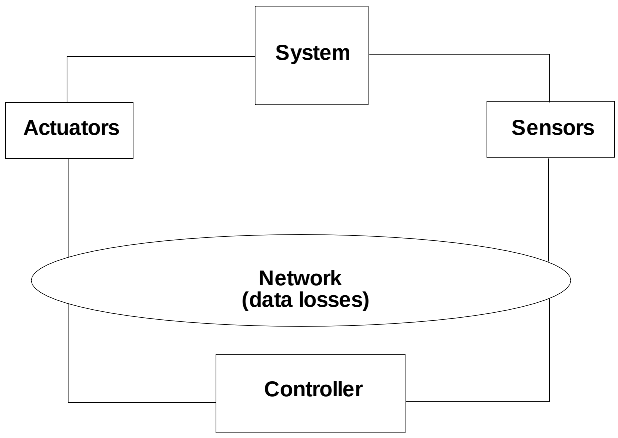

Figure 1: Networked control system with data losses

When the data losses happen, some strategies make the control input zero ("zero control strategy"), or keep the last implemented control input ("last available control") [29]. Instead, in the present paper we use the predictive control scheme to update the control signal. An auxiliary model of the system (1) is introduced as follows:

$$dy(t) = F(y(t), u(t))dt \tag{5}$$

where initial values $y(t_{k_j}) = x(t_{k_j})$. Because there exist data losses when $t \in (t_{k_j}, t_{k_{j+1}})$, we don't know the actual state of $x(t)$. Instead, we use $y(t)$ as an estimation of $x(t)$. By the estimated states $y(t_{k_j} + i\Delta)(i = 0, 1, 2, ..., N_D)$ and the controller $u(t) = h(y(t))$, we generate a sequence of control signals $\left[u(t_{k_j}|t_{k_j}), u(t_{k_j} + \Delta|t_{k_j}), ..., u(t_{k_j} + N_D\Delta|t_{k_j})\right]$ which is packed and transmitted to the actuator, where $u(t_{k_j} + i\Delta|t_{k_j}) = h(y(t_{k_j} + i\Delta))$. When the actuator receives this control sequence, it will implement the first control signal $u(t) = u(t_{k_j}|t_{k_j}),\ t \in [t_{k_j}, t_{k_j} + \Delta)$. If at the sample instant $t_{k_j} + i\Delta(1 \le i \le N_D)$, $\theta(t_{k_j} + i\Delta) = 0$, then the actuator will implement the predictive control signal $u(t) = u(t_{k_j} + i\Delta|t_{k_j}),\ t \in [t_{k_j} + i\Delta, t_{k_j} + (i+1)\Delta)$. When a new control sequence $\left[u(t_{k_{j+1}}|t_{k_{j+1}}), u(t_{k_{j+1}} + i\Delta|t_{k_{j+1}}), ..., u(t_{k_{j+1}} + N_D\Delta|t_{k_{j+1}})\right]$ is received, the actuator will apply the new and the above steps are repeated.

Since $y(t_{k_j}) = x(t_{k_j})$, only the first predictive control signal $u(t_{k_j}|t_{k_j})$ is based on the actual state $x(t)$, and the remaining predictive control signals $u(t_{k_j} + i\Delta|t_{k_j})(1 \le i \le N_D)$ are all based on the estimated state $y(t)$. Then the important question is whether the stability can be guaranteed by the present scheme. In order to answer this question, the following discussion is divided into three sections. The first is about the stability properties of system (1) for $t \in [t_{k_j}, t_{k_j} + \Delta)$. The second is the error estimation between (1) and (5). The third is concerned with the stability properties of the system (5) for $t \in [t_{k_j} + i\Delta, t_{k_j} + (i+1)\Delta)$. Together with these results, the stability of the system (1) in presence of data losses is finally discussed.

Lemma 1 *For the system (1), there exists the constant* $\gamma = \sqrt{2(M_1^2\Delta + M_2^2)}$ *such that the following inequality holds*

$$E\left\|x(t) - x(t_{k_j})\right\| \leq \gamma\sqrt{\Delta}, t \in [t_{k_j}, t_{k_j} + \Delta). \tag{6}$$

Remark 3 *Similar to the above result, for the system (5) there also exists* $\mu = M_1$ *such that the following inequality holds:*

$$E||y(t) - y(t_{k_j} + i\Delta)|| \leq \mu\Delta, \forall t \in [t_{k_j} + i\Delta, t_{k_j} + (i+1)\Delta) \subset [t_{k_j}, t_{k_{j+1}}) \tag{7}$$

Lemma 2 *For the candidate Lyapunov function* $V(x(t))$*,* $EV(x(t))$*is continuous on* $t \geq 0$*.*

A similar proof of the above lemma can be found in [30], and it is omitted here.

Theorem 1 *The system (1) under the control input* $u(t) = h(x(t_{k_j}))$ *can be rewritten as:*

$$dx(t) = F(x(t), h(x(t_{k_j})))dt + l(x(t))\Sigma d\omega(t),\ t \in [t_{k_j}, t_{k_j} + \Delta)$$

Then for any $EV(x(t_{k_j})) \leq r_1 (r_1 > 0)$*, there exists small constants* $\Delta^* > 0$ *and* $\delta > 0$ *such that when* $\Delta \in (0, \Delta^*]$ *the following inequalities hold:*

$$EV(x(t)) \leq \max\left\{EV(x(t_{k_j})),\ r_{\min}\right\},\ \forall t \in [t_{k_j}, t_{k_j} + \Delta)$$

$$EV(x(t_{k_j} + \Delta)) \leq \max\left\{EV(x(t_{k_j})) - \delta\Delta,\ r_{\min}\right\}$$

where $r_{\min} = \max\limits_{\Delta_1 \in [0,\Delta]} \{EV(x(t + \Delta_1)) : EV(x(t)) \leq r_2\}$ *and* $r_2 = (\delta + \gamma\eta_\Psi\sqrt{\Delta})\big/\rho$*.*

Lemma 3 *For the system (1) and the system (5) with the same initial condition* $y(t_{k_j}) = x(t_{k_j})$*, the following inequality holds:*

$$E\left\|x(t) - y(t)\right\| \leq \sqrt{\frac{\alpha}{\beta}(\exp(\beta(t - t_{k_j})) - 1)}, t \in [t_{k_j}, t_{k_{j+1}}) \tag{8}$$

where $\alpha = 2M_2^2, \beta = 2N_D\Delta k_1^2$ *.*

Lemma 4 *If we define two functions* $s(t) = \lambda\sqrt{\frac{\alpha}{\beta}(\exp(\beta t) - 1)}$ *and* $v(t) = \varepsilon t,\ t \geq 0$*, then they have the following properties:*

(1) $s(t)$ *is a strictly increasing function.*

(2) $s(t)$ *has an inflection point* $(\frac{1}{\beta}\ln^2, \lambda\sqrt{\frac{\alpha}{\beta}})$ *such that it is concave for* $t \leq \frac{1}{\beta}\ln^2$ *and convex for* $t \geq \frac{1}{\beta}\ln^2$*.*

(3) For a small $\Delta^{**} > 0$*, when* $\Delta \in (0, \Delta^{**}]$ *and* $\varepsilon \geq \varepsilon^* = \lambda\sqrt{\frac{\alpha}{\beta}(\exp(\beta\Delta) - 1)}\big/\Delta$*,* $s(t)$ *and* $v(t)$ *will have three intersection points* $(0,0), (\bar{t}_1, \bar{s}_1), (\bar{t}_2, \bar{s}_2)$*. Moreover, if* $\varepsilon = \varepsilon^*$*,* $(\bar{t}_1, \bar{s}_1) = (\Delta, \lambda\sqrt{\frac{\alpha}{\beta}(\exp(\beta\Delta) - 1)})$*.*

(4) When $t \in (\bar{t}_1, \bar{t}_2)$ *,* $v(t) > s(t)$ *and when* $t \in [0, \bar{t}_1] \cap [\bar{t}_2, \infty)$ *,* $v(t) \leq s(t)$*.*

The proof of the above lemma is easy and it is omitted here. Lemma 4 will be used in the proof of Theorem 2.

Lemma 5 *Consider the sampled trajectory $y(t)$ of the system (5) under the control law $u = h(y(t))$, which satisfies the conditions of Assumption 1 and is implemented in a sample-and-hold fashion:*

$$\dot{y}(t) = F(y(t), u(t)),\ t \in [t_{k_j}, t_{k_{j+1}}) \tag{9}$$

where initial state $y(t_{k_j}) = x(t_{k_j})$ is a random vector. Then there exists a small $\Delta^{***} > 0$ where $\Delta^{***} \leq \Delta^{**}$ so that the following inequalities hold for any $\Delta \in (0, \Delta^{***}]$

$$EV(y(t)) \leq \max\left\{EV(y(t_{k_j} + i\Delta)),\ \bar{r}_{\min}\right\}, \forall t \in [t_{k_j} + i\Delta, t_{k_j} + (i+1)\Delta) \subset [t_{k_j}, t_{k_{j+1}})(1 \leq i \leq N_D) \tag{10}$$

$$EV(y(t_{k_j} + i\Delta)) \leq \max\left\{EV(y(t_{k_j})) - i\varepsilon\Delta,\ \bar{r}_{\min}\right\}. \tag{11}$$

where $\bar{r}_{\min} = \max_{\Delta_1 \in [0,\Delta]} \{EV(y(t+\Delta_1)) : EV(y(t)) \leq r_3\}$, $r_3 = (\varepsilon + \mu\eta_\varphi\Delta)/\rho$ and $\varepsilon \geq \varepsilon^* = \lambda\sqrt{\frac{\alpha}{\beta}(\exp(\beta\Delta) - 1)}\Big/\Delta$.

Remark 4 *By Theorem 1 and Lemma 5, for a fixed sampling time Δ, $r_{\min}$ and $\bar{r}_{\min}$ decrease as r_2 and r_3 decrease, respectively. Moreover, both r_2 and r_3 decrease with the increasing of ρ. This implies that $r_{\min}$ and $\bar{r}_{\min}$ can be small enough by changing the value of ρ. One the other hand, if $F(y, u) = f(y) + g(y)u$ and the controller (4) is used, the time derivative of $V(y(t))$ includes the term $-\sqrt{\psi^2 + (L_g V (L_g V)^T)^2}$ which also contributes to the convergence of the system. Therefore, in most situations $r_{\min}$ and $\bar{r}_{\min}$ can be small enough by using a proper ρ.*

Theorem 2 *Consider the system (1) with the controller $h(x)$ at asynchronous time instants $\{t_{k_j}\}$. For $x(t_0) \in \Omega_{r_1}$, if $\Delta \leq \max\{\Delta^*, \Delta^{**}, \Delta^{***}\}$, $\varepsilon \geq \varepsilon^* = \lambda\sqrt{\frac{\alpha}{\beta}(\exp(\beta\Delta) - 1)}\Big/\Delta$ and*

$$-N_D\varepsilon\Delta + \lambda\sqrt{\frac{\alpha}{\beta}(\exp(\beta N_D\Delta) - 1)} < 0$$

then there exists a constant $R_{\min} > 0$ such that the following inequality holds:

$$\lim_{t\to\infty} \sup EV(x(t)) \leq R_{\min} \tag{12}$$

where $R_{\min} = \max\{r_{\min}, \bar{r}_{\min} + \lambda\sqrt{\frac{\alpha}{\beta}(\exp(\beta N_D\Delta) - 1)}\}$.

Theorem 3 *If there exist constants $c_1, c_2 > 0$ satisfying*

$$c_1||x(t)||^p \leq V(x(t)) \leq c_2||x(t)||^p, (p > 0) \tag{13}$$

then

$$\lim_{t\to\infty} \sup E||x(t)||^p \leq R_{\min}/c_1 \tag{14}$$

The proof of (14) is easy and it is omitted here. The formula (14) shows that the pth moments of the trajectories are eventually not beyond the $\frac{R_{\min}}{c_1}$-neighbourhood of the origin.

Remark 5 *From the definition of $R_{\min} = \max\{r_{\min}, \bar{r}_{\min} + \lambda\sqrt{\frac{\alpha}{\beta}(\exp(\beta N_D\Delta) - 1)}\}$, the value of $R_{\min}$ is dependent on $r_{\min}, \bar{r}_{\min}$, and $\lambda\sqrt{\frac{\alpha}{\beta}(\exp(\beta N_D\Delta) - 1)}$. According to the discussion of Remark 4, $r_{\min}$ and $\bar{r}_{\min}$ can make small enough by tuning the value of ρ, therefore under a previously given controller $R_{\min}$ is* *mainly affected by the value of* $\lambda\sqrt{\frac{\alpha}{\beta}(\exp(\beta N_D\Delta)-1)}$. *On the other hand, if the control target is to make the pth moment of the system eventually be in a small* η*-neighbourhood of the origin, namely*

$$\lim_{t\to\infty}\sup E||x(t)||^p\le\eta, \tag{15}$$

then by the above discussions and (14), a sufficient condition can be taken as

$$N_D\Delta\le\frac{1}{\beta}\ln[\frac{\beta}{\alpha}(\frac{c_1\eta-\pi}{\lambda})^2+1], \tag{16}$$

where $\pi=\max\{r_{\min},\bar{r}_{\min}\}$. *This implies the maximum time of consecutive packet dropouts should be no more than the above upper boundedness in order to achieve the expected control target.*

Remark 6 *The proposed scheme needs to generate a series of predictive control signals when the new state information is available, which seems to be more complicated than the zero control and the last available control in dealing with packet losses. However, this scheme doesn't need to estimate in advance the probability distribution of packet losses that is the concern of several others, but needs to know roughly an upper bound of consecutive packet losses. Thus, it has little limitation and is easy to implement in practical application. On the other hand, common predictive control schemes generally involve the online optimization of controlled plants. For nonlinear systems, it will cost a lot of computation time, and the obtained numerical solution is not always optimal, but often sub-optimal. In order to avoid this difficulty, for a large class of controlled systems which has been described in Remark 1, the Lyapunov-based controller (4) with inverse optimality is introduced to improve the real-time performance and the online computing efficiency. Consequently, the proposed method is high-efficient and applicable widely.*

4 Numerical example

The famous Lorenz system can be written as [31]:

$$\begin{cases}\dot{x}_1=a(x_2-x_1)\\ \dot{x}_2=bx_1-x_2-x_1x_3\\ \dot{x}_3=-cx_3+x_1x_2\end{cases} \tag{17}$$

In this paper, we suppose the parameters a,b,c are with random disturbances and $a=10+\sigma_1\xi_1(t), b=28+\sigma_2\xi_2(t), c=\frac{8}{3}+\sigma_3\xi_3(t)$, where $\sigma_i(i=1,2,3)$ denote the noise intensity, and $\xi_i(t)(i=1,2,3)$ are mutually independent standard Gaussian white noise which can be expressed as the formal derivative of Wiener processes $\omega_i(t)(i=1,2,3)$, namely $\xi_i(t)=d\omega_i(t)/dt(i=1,2,3)$. It is obvious that the origin $(0,0,0)$ is an equilibrium point of the system. Now let us consider the networked predictive control problem of the system (17), and the controlled Lorenz system is as follows

$$dx=f(x)dt+l(x)\Sigma d\omega(t)+Bu(t)dt \tag{18}$$

where

$$x=\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix},\quad f(x)=\begin{pmatrix}10(x_2-x_1)\\ 28x_1-x_2-x_1x_3\\ -\frac{8}{3}x_3+x_1x_2\end{pmatrix}$$

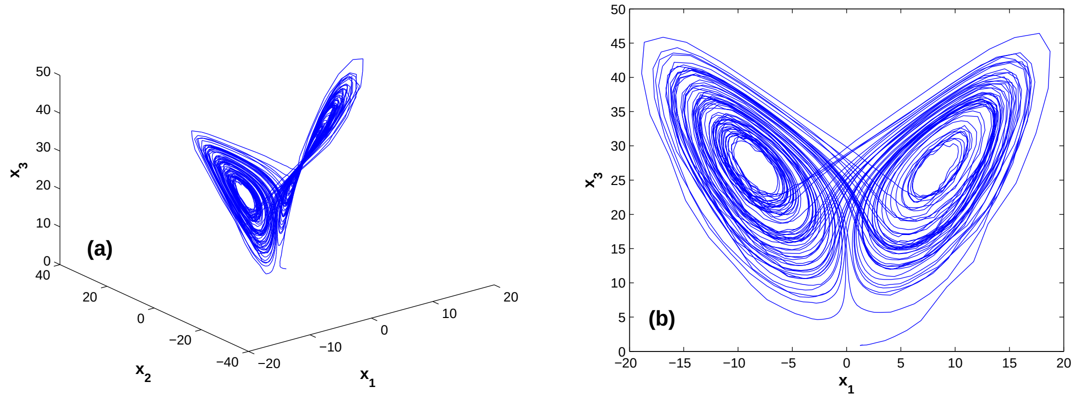

Figure 2: (a) phase portrait in (x_1, x_2, x_3) space, (b) projective portrait in (x_1, x_3) plane

$$l(x(t))\Sigma d\omega(t) = \begin{pmatrix} \sigma_1(x_2 - x_1) & 0 & 0 \\ 0 & \sigma_2 x_1 & 0 \\ 0 & 0 & -\sigma_3 x_3 \end{pmatrix} \begin{pmatrix} d\omega_1(t) \\ d\omega_2(t) \\ d\omega_3(t) \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

In order to apply the proposed method, we choose the Lyapunov function $V(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2)$, and by the formula (4) the controller is taken as

$$u = h(x) = \begin{cases} -\frac{\omega(x)+\sqrt{\omega(x)^2+x_2^4}}{x_2}, x_2 \neq 0 \\ 0 \qquad\qquad\qquad\quad , x_2 = 0 \end{cases} \tag{19}$$

where

$$\omega(x) = (38x_1x_2 - x_2^2 - 10x_1^2 - \frac{8}{3}x_3^2) + \frac{1}{2}(\sigma_1^2(x_2 - x_1)^2 + \sigma_2^2 x_1^2 + \sigma_3^2 x_3^2) + V(x)$$

The auxiliary predictive model is

$$dy = f(y)dt + Bu(t)dt \tag{20}$$

At each sampling instant t_{k_j}, by the predictive model (20) and the controller $u(t) = h(y(t))$, we generate a sequence of control signals $\left[u(t_{k_j}|t_{k_j}), u(t_{k_j} + \Delta|t_{k_j}), ..., u(t_{k_j} + N_D\Delta|t_{k_j})\right]$ which is packed and transmitted to the actuator, where $u(t_{k_j} + i\Delta|t_{k_j}) = h(y(t_{k_j} + i\Delta)), i = 0, 1, ...N_D$.

To generate the increasingly random sequence of times $\{t_{k_j}\}$, we take $k_{j+1} - k_j$ to be random integer with the uniform distribution on $[1, N_D]$. In numerical simulations, we take the initial value $x(0) = [2, -1, 1]^T$, the noise intensity $\sigma_1 = \sigma_2 = \sigma_3 = 0.2, \Delta = 0.01, N_D = 10, T = N_D\Delta$, and use the runge-kutta method with a fixed time step 0.01. Fig.2 displays the phase portrait of stochastically chaotic Lorenz system (17). When the predictive control is applied to the chaotic system, from Figs. 3-4 one can see that the state trajectories of the chaotic system and the control action will approach to the neighborhoods of zero points as time increases. These results investigate the effectiveness of the proposed method for the chaotic systems with stochastic disturbances and data losses.

On the other hand, we also compare the present method with the other two methods, one is the last available control strategy with the controller applied in a sample-and-hold fashion, and the other is the zero control strategy. Three control methods are all applied in the worst case where the system receives

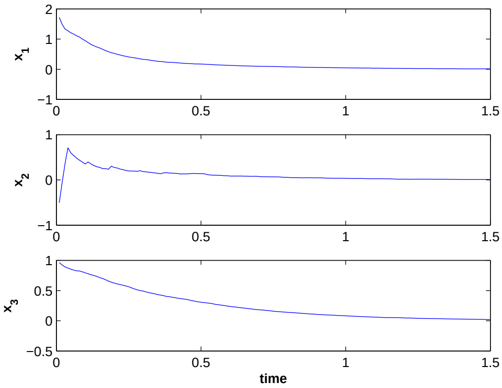

Figure 3: The evolution of system states under the predictive control

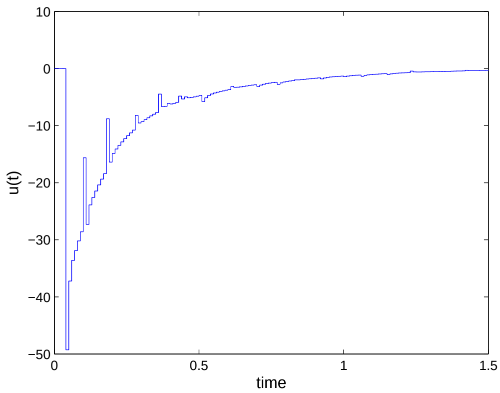

Figure 4: The varying curve of control input by the predictive control

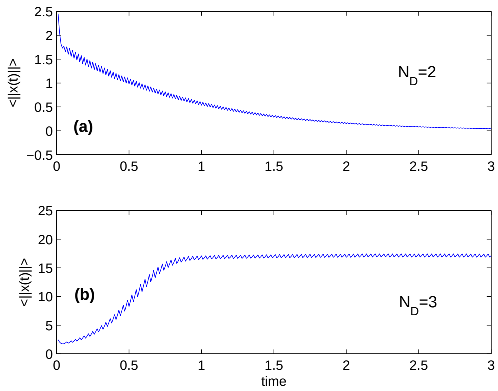

Figure 5: The control results of zero control input when data loss happen (a)$N_D = 2$, (b) $N_D = 3$

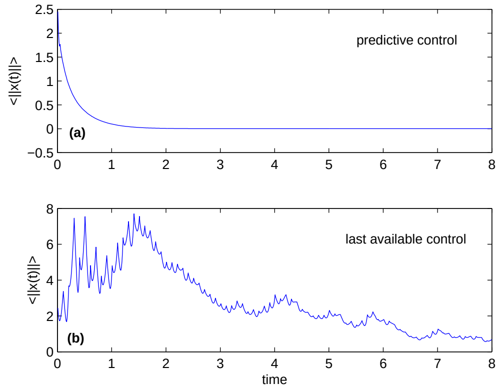

Figure 6: The control results of different strategies when the noise intensities $\sigma_1 = \sigma_2 = \sigma_3 = 0.2$ and the maximum time of data packet losses $N_D = 10$ (a) the proposed predictive control, (b)the last available control

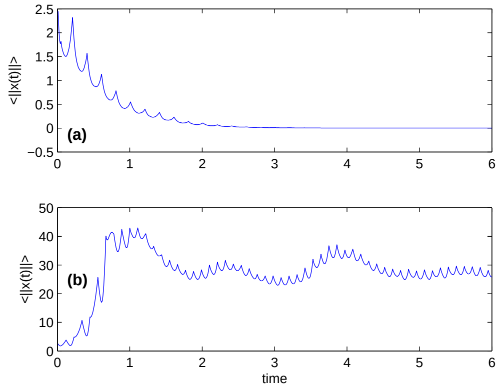

Figure 7: The control results of different strategies when data loss happen (a) the proposed predictive control when $\sigma_1 = \sigma_2 = \sigma_3 = 0.8$ and $N_D = 20$, (b) the last available control when $\sigma_1 = \sigma_2 = \sigma_3 = 0.2$ and $N_D = 11$

only one measurement of the actual state every N_D sampling time, namely $k_{j+1} - k_j = N_D(j = 1, 2, ...)$. In order to describe the control effectiveness, the index $< ||x(t)|| >$ is introduced and defined as the average value of $||x(t)||$ over 100 stochastic simulations. When $\sigma_1 = \sigma_2 = \sigma_3 = 0.2$, the results of the zero control on $N_D = 2$ and $N_D = 3$ are shown in Fig. 5. From Fig. 5, we can see that when data packets are lost the zero control strategy can only stabilize the system for the small N_D. When $\sigma_1 = \sigma_2 = \sigma_3 = 0.2$ and $N_D = 10$, Fig. 6(a)and 6(b) give the results of the predictive control and the last available control, respectively. From Fig. 6(a),(b) we can see that the other two control strategies are better than the zero control, and the predictive control gives the better results than the last available control. Fig.7 displays the different control results when the noise intensity and the maximum time of data losses increase. It can be found that with the increasing of noise intensity and the maximum time of data losses, the predictive control can tolerate the higher noise intensity and the longer time of data losses, which implies that the proposed predictive control method is more robust than the other two control methods for stochastic nonlinear sytems.

5 Conclusions

In this paper the networked control problem of nonlinear systems with stochastic disturbances in presence of data packet losses is investigated. Based on the model predictive control and a Lyapunov-based controller, the compensation technique is presented and a sequence of predictive control signals is generated so that the system can update the control input when the system information is not available. Together with stochastic stability theory, the practical stability of NCS is discussed in detail. It shows that when the time of consecutive data losses is less than some reasonable boundness, the pth moment of the system will eventually remain inside a neighborhood of the origin. Numerical simulations and comparisons with the last available control and the zero control are carried out. It is found that the proposed method can tolerate the higher noise intensity and the longer time of packet losses than the other two schemes, which indicates the effectiveness of the proposed method. Further works include the extension of the proposed method to stochastic nonlinear systems with network-induced delays, which may get inspiration from [32] based on the method of Markovian jump sytems with delays. Also, because fuzzy models are able to approximate any smooth nonlinear functions to any degree of accuracy [33,34], it would be interesting to consider the networked control of nonlinear systems based on fuzzy models by using the proposed scheme in the future.

Acknowledgments

The authors gratefully acknowledge the financial support of the National Natural Science Foundation of China (Grant Nos.11572231 and 11202155),the Education Department Foundation of Shaanxi(Grant No. 2013JK0595), the Natural Science Foundation of Shaanxi(Grant No. 2014JQ9372) and the Statistical Bureau Foundation of China(Grant No. 2013LY067).

References

[1] W. Zhang, M. S. Branicky, S. M. Phillips, Stability of networked control systems, IEEE Control Syst. Mag. 21(2001) 84-89.

[2] T. C. Yang, Networked control systems: A brief survey, Proc. Inst. Elect. Eng., 153(2006) 403-412.

[3] G.C. Walsh, H. Ye, Scheduling of networked control systems, IEEE Control Systems Magazine,21(1)(2001)57-65.

[4] R. Yang, G. P. Liu, P. Shi, C. Thomas. Predictive output feedback control for networked control systems. Industrial Electronics, IEEE Transactions on, 61(1) (2014) 512-520.

[5] L. Zhang, H. Gao,O. Kaynak. Network-induced constraints in networked control systemsa survey. IEEE Transactions on Industrial Informatics, 9(1)(2013) 403-416.

[6] Q. Shafiee ,C. Stefanovic ,T. Dragicevic, et al. Robust networked control scheme for distributed secondary control of islanded microgrids. IEEE Transactions onIndustrial Electronics, 61(10)(2014) 5363-5374.

[7] Z. Chen,B. Zhang, H. Li, J. Yu, Tracking control for polynomial fuzzy networked systems with repeated scalar nonlinearities. Neurocomputing, 171(1) (2016) 185-193.

[8] O. C. Imer, S. Yksel, T. Basar, Optimal control of LTI systems over unreliable communications links, Automatica,42(2006) 1429-1439.

[9] M. Moayedi, Y. K. Foo,Y. C. Soh, LQG Control for Networked Control Systems with Random Packet Delays and Dropouts via Multiple Predictive-Input Control Packets, Preprints of the 18th IFAC World Congress Milano,Italy, 2011, pp. 72-77.

[10] Z. Xiang, X. Jian, Communication and Control Co-Design for Networked Control System in Optimal Control, Proc. of the 12th WSEAS International Conference on SYSTEMS, Heraklion, Greece, 2008, pp. 698-703.

[11] M. Yu, L. Wang, T. Chu,G. Xie, Stabilization of Networked Control Systems with Data Packet Dropout and Network Delays via Switching System Approach, 43rd IEEE Conference on Decision and Control, Atlantis, Paradise Island, Bahamas,2004,pp.3539-3544.

[12] J. Xiong, J. Lam, Stabilization of linear systems over networks with bounded packet loss, Automatica, 43(1)(2007) 80-87.

[13] L. A. Montestruque,P. J. Antsaklis, On the model-based control of networked systems, Automatica, 39(2003)1837-1843.

[14] L. A. Montestruque, P. J. Antsaklis, Stability of model-based networked control systems with time-varying transmission times, IEEE Trans. Automat. Control, 49(9)(2004)1562-1572.

[15] Gilberto Pin, Thomas Parisini, Networked Predictive Control of Uncertain Constrained Nonlinear Systems: Recursive Feasibility and Input-to-State Stability Analysis, IEEE Transactions on Automatic Control, 56(1)(2011)72-87.

[16] D. Munoz de la Pena, P. D. Christofides, Lyapunov-Based Model Predictive Control of Nonlinear Systems Subject to Data Losses, IEEE Trans. Automat. Control, 53(9)(2008)2076-2089.

[17] Y. B. Zhao, G. P. Liu,D. Rees, A predictive control based approach to networked Hammerstein systems: Design and stability analysis, IEEE Trans. Syst. Man Cybern-Part B: Cybem., 38(3)(2008)700-708.

[18] G.P. Liu, Predictive controller design of networked systems with communication delays and data loss, IEEE Transactions on Circuits and Systems II, 57(6)(2010)481-485.

[19] J. F. Liu, X. Z. Chen, D. Munoz de la Pena, P. D. Christofides, Iterative Distributed Model Predictive Control of Nonlinear Systems: Handling Asynchronous, Delayed Measurements,IEEE Trans. Automat. Control, 57(2)(2012)528-534.

[20] M. Mahmood, P. Mhaskar,Lyapunov-based model predictive control of stochastic nonlinear systems,Automatica, 48(9)(2012)2271-2276.

[21] B. Stefano, A. D. Santis, Stabilization in probability of nonlinear stochastic systems with guaranteed region of attraction and target set. IEEE Transactions on Automatic Control, 48(9)(2003) 1585-1599.

[22] J. Hu, Z. Wang, B. Shen, H. Gao, Quantised recursive filtering for a class of nonlinear systems with multiplicative noises and missing measurements, International Journal of Control 86(4)(2013)650-663.

[23] Z. Wang, B. Shen, H. Shu, G. Wei, Quantized control for nonlinear stochastic time-delay systems with missing measurements, IEEE Transactions on Automatic Control, 57(6)(2012)1431-1444.

[24] E. Sontag, A 'universal' construction of arstein's theorem on nonlinear stabilization, Syst. Control Lett., 13(1989)117-123.

[25] H. Deng, M. Krstic, Stochastic nonlinear stabilization-II: Inverse optimality, Syst. Control Lett., 32(1997)151-159.

[26] J. A. Primbs, V. Nevistic, J. C. Doyle. Nonlinear optimal control: A control Lyapunov function and receding horizon perspective. Asian Journal of Control,1(1)(1999) 14-24.

[27] G.P. Liu, J. Mu, D. Rees, Networked predictive control of systems with random communication delay, Proceedings of the UKACC Control, Bath, 2004.

[28] G.P. Liu, D. Rees, S. C. Chai, X. Y. Nie, Design simulation and implementation of networked predictive control systems, Measurement and Control, 38(1)(2005)17-21.

[29] D. Munoz de la Pena, P. D. Christofides, Model-Based Control of Nonlinear Systems Subject to Sensor Data Losses: A Chemical Process Case Study, Proceedings of the 46th IEEE Conference on Decision and Control, New Orleans, LA, USA, 2007, pp.3333-3338.

[30] L.R. Huang,X.R. Mao, On Input-to-State Stability of Stochastic Retarded Systems With Markovian Switching, IEEE Trans. Automat. Control, 54(8)(2009) 1898-1902.

[31] E.N. Lorenz, Deterministic non-periodic flow, Journal of Atmospheric Sciences, 20(2)(1963)130-141.

[32] J. Qiu, Y. Wei,H. R. Karimi, New approach to delay-dependent H_∞ control for continuous-time Markovian jump systems with time-varying delay and deficient transition descriptions, Journal of the Franklin Institute, 352(1)(2015)189-215.

[33] J. Qiu,G. Feng,H. Gao, Static-Output-Feedback Control of Continuous-Time T-S Fuzzy Affine Systems Via Piecewise Lyapunov Functions, IEEE Transactions on Fuzzy Systems, 21(2)(2013) 245-261.

[34] J. Qiu, H. Tian,Q. Lu, H. Gao. Nonsynchronized robust filtering design for continuous-time TCS fuzzy affine dynamic systems based on piecewise Lyapunov functions, IEEE Transactions onCybernetics, 43(6)(2013)1755-1766.

Appendix

.A Proof of Lemma 1

Integrating the system (1) from t_{k_j} to t, we get

$$x(t) - x(t_{k_j}) = \int_{t_{k_j}}^{t} F(x(\tau), u(\tau))d\tau + \int_{t_{k_j}}^{t} l(x(\tau))\Sigma d\omega(t),\ t \in [t_{k_j}, t_{k_j} + \Delta)$$

Using Cauchy-Schwarz inequality and Assumption 1, we can obtain

$$E\left(\left\|\int_{t_k}^{t} F(x(\tau), u(\tau))d\tau\right\|^2\right) \leq (t - t_k)E\left(\int_{t_k}^{t} \|F(x(\tau), u(\tau))\|^2 d\tau\right) \leq M_1^2(t - t_k)^2 \leq M_1^2\Delta^2$$

$$E\left(\left\|\int_{t_k}^{t} l(x(\tau))\Sigma d\omega(t)\right\|^2\right) = E\left(\int_{t_k}^{t} \|l(x(\tau))\Sigma\|^2 d\tau\right) \leq M_2^2(t - t_k) \leq M_2^2\Delta$$

Thus, we get

$$E\left(\|x(t) - x(t_{k_j})\|^2\right) \leq 2(M_1^2\Delta + M_2^2)\Delta$$

If we take $\gamma^2 = 2(M_1^2\Delta + M_2^2)$, then Jensen's inequality implies that

$$E\|x(t) - x(t_{k_j})\| = E\sqrt{\|x(t) - x(t_{k_j})\|^2} \leq \sqrt{E\left(\|x(t) - x(t_{k_j})\|^2\right)} \leq \gamma\sqrt{\Delta}$$

.B Proof of Theorem 1

The time derivative of the Lyapunov function $V(t)$ along the trajectory $x(t)$ of the system (1) in $t \in [t_{k_j}, t_{k_j} + \Delta)$ is given by

$$\dot{V}(x(t)) = \Psi(x(t), h(x(t_{k_j}))) + \frac{\partial V(x(t))}{\partial x} l(x(t))\Sigma d\omega(t)$$

Adding and subtracting $\Psi(x(t_{k_j}), h(x(t_{k_j})))$, and taking into account Assumption 1, we can obtain

$$\begin{aligned}\dot{V}(x(t)) &\leq -\rho V(x(t_{k_j})) + \Psi(x(t), h(x(t_{k_j}))) - \Psi(x(t_{k_j}), h(x(t_{k_j}))) + \tfrac{\partial V(x(t))}{\partial x} l(x(t))\Sigma d\omega(t)\\ &\leq -\rho V(x(t_{k_j})) + \eta_\Psi ||x(t) - x(t_{k_j})|| + \tfrac{\partial V(x(t))}{\partial x} l(x(t))\Sigma d\omega(t)\end{aligned}$$

Taking the expectation of the above inequality and using Lemma 1, it leads to

$$E\dot{V}(x(t)) \leq -\rho EV(x(t_{k_j})) + \eta_\Psi E||x(t) - x(t_{k_j})|| \leq -\rho EV(x(t_{k_j})) + \gamma\eta_\Psi\sqrt{\Delta} \tag{21}$$

where Δ is the sampling time, and the value of $\gamma\eta_\Psi\sqrt{\Delta}$ can be arbitrarily small provided that Δ is sufficiently small. Therefore, there exists a small $\Delta^* > 0$ such that when $\Delta \in (0, \Delta^*]$ the following properties holds:

(i) Take proper constants $\delta > 0, \rho > 0$, such that $r_2 = (\delta + \gamma\eta_\Psi\sqrt{\Delta})\big/\rho$ and $r_2 < r_1$. When $r_2 < EV(x(t_{k_j})) \leq r_1$, then by (21) $E\dot{V}(x(t)) \leq -\delta$. Integrating this inequality, we obtain $EV(x(t)) \leq EV(x(t_{k_j})) - (t - t_{k_j})\delta, \forall t \in [t_{k_j}, t_{k_j} + \Delta)$.

(ii) If let $r_{\min} = \max\limits_{\Delta_1 \in [0,\Delta]} \{EV(x(t + \Delta_1)) : EV(x(t)) \leq r_2\}$, then we can always take a number Δ^* small enough such that $r_{\min} < r_1$. Thus, when $EV(x(t_{k_j})) \leq r_2$, by the definition of $r_{\min}$ one knows $EV(x(t)) \leq r_{\min}, \forall t \in [t_{k_j}, t_{k_j} + \Delta)$.

Through the above discussion, for any $EV(x(t_{k_j})) \leq r_1$ we obtain

$$EV(x(t)) \leq \max\left\{EV(x(t_{k_j})) - (t - t_{k_j})\delta,\ r_{\min}\right\}, \forall t \in [t_{k_j}, t_{k_j} + \Delta) \tag{22}$$

The inequality (22) further implies

$$EV(x(t)) \leq \max\left\{EV(x(t_{k_j})),\ r_{\min}\right\}, \forall t \in [t_{k_j}, t_{k_j} + \Delta)$$

$$EV(x(t_{k_j} + \Delta)) \leq \max\left\{EV(x(t_{k_j})) - \delta\Delta,\ r_{\min}\right\}$$

.C Proof of Lemma 3

Let the error vector $e(t) = x(t) - y(t)$ and by the systems (1) and (5) we can obtain

$$de(t) = [F(x(t), u) - F(y(t), u)]dt + l(x(t))\Sigma d\omega(t)$$

Integrating the above formula from t_{k_j} to t, we get

$$e(t) = \int_{t_{k_j}}^{t} (F(x(\tau), u) - F(y(\tau), u))d\tau + \int_{t_{k_j}}^{t} l(x(\tau))\Sigma d\omega(\tau)$$

Applying the inequality $\|a + b\|^2 \leq 2||a||^2 + 2||b||^2$, then

$$\|e(t)\|^2 \leq 2\left\|\int_{t_{k_j}}^{t} (F(x(\tau), u) - F(y(\tau), u))d\tau\right\|^2 + 2\left\|\int_{t_{k_j}}^{t} l(x(\tau))\Sigma d\omega(\tau)\right\|^2$$

Taking $T = N_D\Delta$, using Cauchy-Schwarz inequality and considering $t_{k_{j+1}} - t_{k_j} \leq T$

$$\begin{aligned}
&\left\|\int_{t_{k_j}}^{t} (F(x(\tau), u) - F(y(\tau), u))d\tau\right\|^2 \leq (t - t_{k_j}) \int_{t_{k_j}}^{t} \|F(x(\tau), u) - F(y(\tau), u)\|^2 d\tau \\
&\leq Tk_1^2 \int_{t_{k_j}}^{t} \|e(\tau)\|^2 d\tau \\
&E\left\|\int_{t_{k_j}}^{t} l(x(\tau))\Sigma d\omega(\tau)\right\|^2 = E\int_{t_{k_j}}^{t} \|l(x(\tau))\Sigma\|^2 d\tau \leq \int_{t_{k_j}}^{t} M_2^2 d\tau
\end{aligned}$$

then

$$E\|e(t)\|^2 \leq \int_{t_{k_j}}^{t} (2M_2^2 + 2Tk_1^2 E\|e(\tau)\|^2)d\tau$$

Let $\alpha = 2M_2^2, \beta = 2Tk_1^2$, and by the Gronwall Integral Inequality we obtain

$$E\|e(t)\|^2 \leq \frac{\alpha}{\beta}(\exp(\beta(t - t_{k_j})) - 1)$$

By Jensen's inequality

$$E\|e(t)\| \leq \sqrt{\frac{\alpha}{\beta}(\exp(\beta(t - t_{k_j})) - 1)}, t \in [t_{k_j}, t_{k_{j+1}})$$

.D Proof of Lemma 5

Let us consider the time derivative of the Lyapunov function $V(y)$ along the trajectory $y(t)$ of the system (9) in $t \in [t_{k_j} + i\Delta, t_{k_j} + (i+1)\Delta) \subset [t_{k_j}, t_{k_{j+1}})$

$$\dot{V}(y(t)) = L_F V(y(t), h(y(t_{k_j} + i\Delta))) = \varphi(y(t), h(y(t_{k_j} + i\Delta)))$$

Adding and subtracting $\varphi(y(t_{k_j} + i\Delta), h(y(t_{k_j} + i\Delta)))$, and taking into account Assumption 1, we can obtain

$$\begin{aligned}
&\dot{V}(y(t)) \leq \varphi(y(t_{k_j} + i\Delta), h(y(t_{k_j} + i\Delta))) + \varphi(y(t), h(y(t_{k_j} + i\Delta))) - \varphi(y(t_{k_j} + i\Delta), h(y(t_{k_j} + i\Delta))) \\
&\leq -\rho V(y(t_{k_j} + i\Delta)) + \eta_\varphi ||y(t) - y(t_{k_j} + i\Delta)||
\end{aligned}$$

Taking the expectation of the above inequality and using (7) it leads to

$$E\dot{V}(y(t)) \leq -\rho EV(y(t_{k_j} + i\Delta)) + \eta_\varphi E||y(t) - y(t_{k_j} + i\Delta)|| \leq -\rho EV(y(t_{k_j} + i\Delta)) + \mu\eta_\varphi\Delta \qquad (23)$$

where Δ is the sampling time, and the value of $\mu\eta_\varphi\Delta$ can be arbitrarily small provided that Δ is sufficiently small. Therefore, there exist a small $\Delta^{***} > 0$ where $\Delta^{***} \leq \Delta^{**}$ such that when $\Delta \in (0, \Delta^{***}]$ the following results can be obtained:

(I) Take the positive constants $\varepsilon \geq \varepsilon^*$, and $r_3 = (\varepsilon + \mu\eta_\varphi\Delta)/\rho$ such that $r_3 < r_1$. When $r_3 < EV(y(t_{k_j} + i\Delta)) \leq r_1$, by (23) we get $E\dot{V}(y(t)) \leq -\varepsilon$. Integrating this inequality it leads to $EV(y(t)) \leq EV(y(t_{k_j} + i\Delta)) - (t - t_{k_j} - i\Delta)\varepsilon, \forall t \in [t_{k_j} + i\Delta, t_{k_j} + (i+1)\Delta)$.

(II) If let $\bar{r}_{\min} = \max\limits_{\Delta_1 \in [0,\Delta]} \{EV(y(t + \Delta_1)) : EV(y(t)) \leq r_3\}$, then we can always take a small Δ^{***} such that $\bar{r}_{\min} < r_1$. Thus, when $EV(y(t_{k_j} + i\Delta)) \leq r_3$ we know $EV(y(t)) \leq \bar{r}_{\min}$ for $\forall t \in [t_{k_j} + i\Delta, t_{k_j} + (i+1)\Delta)$.

By (I) and (II), we know that if $EV(y(t_{k_j} + i\Delta)) \leq r_1$, then

$$EV(y(t)) \leq \max\left\{EV(y(t_{k_j} + i\Delta)) - (t - t_{k_j} - i\Delta)\varepsilon, \ \bar{r}_{\min}\right\}, \forall t \in [t_{k_j} + i\Delta, t_{k_j} + (i+1)\Delta) \qquad (24)$$

(24) implies (10)holds. Furthermore, by the continuity of $EV(y(t))$ on t and using (24) recursively, we conclude if $EV(y(t_{k_j})) \leq r_1$, then

$$EV(y(t_{k_j} + i\Delta)) \leq \max\left\{EV(y(t_{k_j})) - i\varepsilon\Delta, \ \bar{r}_{\min}\right\}$$

.E Proof of Theorem 2

We suppose $t_{k_{j+1}} = t_{k_j} + N_j\Delta, N_j \geq 1$, and the following discussion is divided into two parts:

(a) If $t_{k_{j+1}} = t_{k_j} + N_j\Delta, N_j > 1$, by Lemma 5 we get

$$EV(y(t_{k_{j+1}})) \leq \max\left\{EV(y(t_{k_j})) - N_j\varepsilon\Delta, \ \bar{r}_{\min}\right\} \qquad (25)$$

On the other hand, by Assumption 1 one can obtain

$$V(x(t_{k_{j+1}})) \leq V(y(t_{k_{j+1}})) + f_V(||x(t_{k_{j+1}}) - y(t_{k_{j+1}})||)$$

Taking the expectation of the above inequality, it leads to

$$EV(x(t_{k_{j+1}})) \leq EV(y(t_{k_{j+1}})) + Ef_V(||x(t_{k_{j+1}}) - y(t_{k_{j+1}})||) \qquad (26)$$

By (25),(26) and Lemma 3, we obtain

$$EV(x(t_{k_{j+1}})) \leq \max\left\{EV(x(t_{k_j})) - N_j\varepsilon\Delta,\ \bar{r}_{\min}\right\} + \lambda\sqrt{\frac{\alpha}{\beta}(\exp(\beta N_j\Delta) - 1)} \tag{27}$$

When $\Delta \leq \max\{\Delta^*, \Delta^{**}, \Delta^{***}\}$, $\varepsilon \geq \varepsilon^*$ and $-N_D\varepsilon\Delta + \lambda\sqrt{\frac{\alpha}{\beta}(\exp(\beta N_D\Delta) - 1)} < 0$, by Remark 4 and $1 < N_j \leq N_D$, we can get $-N_j\varepsilon\Delta + \lambda\sqrt{\frac{\alpha}{\beta}(\exp(\beta N_j\Delta) - 1)} < 0$, which implies there exists $\varepsilon_j > 0$ such that the following inequality holds:

$$-N_j\varepsilon\Delta + \lambda\sqrt{\frac{\alpha}{\beta}(\exp(\beta N_j\Delta) - 1)} \leq -\varepsilon_j < 0 \tag{28}$$

If we take $r_D = \bar{r}_{\min} + \lambda\sqrt{\frac{\alpha}{\beta}(\exp(\beta N_D\Delta) - 1)}$ and $R_{\min} = \max\{r_{\min}, r_D\}$, then (27) can be rewritten as

$$EV(x(t_{k_{j+1}})) \leq \max\left\{EV(x(t_{k_j})) - \varepsilon_j,\ r_D\right\} \leq \max\left\{EV(x(t_{k_j})) - \varepsilon_j,\ R_{\min}\right\} \tag{29}$$

When $t \in [t_{k_j} + \Delta, t_{k_{j+1}})$, using Remark 4 we can obtain

$$EV(x(t)) \leq \max\left\{EV(x(t_{k_j})),\ r_D\right\} \leq \max\left\{EV(x(t_{k_j})),\ R_{\min}\right\} \tag{30}$$

When $t \in [t_{k_j}, t_{k_j} + \Delta)$, by Theorem 1 we get

$$EV(x(t)) \leq \max\left\{EV(x(t_{k_j})),\ r_{\min}\right\} \leq \max\left\{EV(x(t_{k_j})),\ R_{\min}\right\} \tag{31}$$

From (30),(31)

$$EV(x(t)) \leq \max\left\{EV(x(t_{k_j})),\ R_{\min}\right\}, \forall t \in [t_{k_j}, t_{k_{j+1}}) \tag{32}$$

(b) If $t_{k_{j+1}} = t_{k_j} + \Delta$, by Theorem 1

$$EV(x(t_{k_{j+1}})) \leq \max\left\{EV(x(t_{k_j})) - \delta\Delta,\ r_{\min}\right\} \leq \max\left\{EV(x(t_{k_j})) - \delta\Delta,\ R_{\min}\right\} \tag{33}$$

$$EV(x(t)) \leq \max\left\{EV(x(t_{k_j})),\ r_{\min}\right\} \leq \max\left\{EV(x(t_{k_j})),\ R_{\min}\right\},\ \forall t \in [t_{k_j}, t_{k_j} + \Delta) \tag{34}$$

Thus, using (29)(32) and (33)(34) recursively, we can conclude if $x(t_0) \in \Omega_{r_1}$ then

$$\lim_{t\to\infty} \sup EV(x(t)) \leq R_{\min}$$